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Iterative Design of Economic Models via Simulation, Optimization and Modeling

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Abstract

Microeconomic models, e.g., concern models, usually suffer from too simple dynamic equations and from too unrealistic economic data. We investigate a new iterative design method, including numerical simulation, numerical solution of optimal control problems by direct optimization methods and modeling. The capability of the method is demonstrated by the refinement of a quite simple first concern model. In the end, very complex micro- and macroecononomic phenomenona known from reality, have been obtained for various numerical calculations.

Introduction and first concern model

Mathematical models of microeconomic processes are very important for many purposes. These models can, e.g., help to explain macroeconomic phenomenona or help to improve the mangement of a concern. Especially concern models are well known in literature, see, e.g., Feichtinger and Hartl (1986), Hilten et al. (1993), Kamien and Schwartz (1981), Kort (1989), Kort et al. (1991) and Lesourne and Leban (1978). The proposed concern models generally can be (and should be!) improved by an additional refinement. Usual insufficiencies are:

- Quite simple dynamic equations to enable an analytic calculation of the optimal open-loop or feedback controls or/and the optimal limit cycles.
- Inexact economic data and inexact data of the concern due to the difficulties to get and to use these data for numerical calculations.
- Fuzzy definition of the business policy due to different possible management approaches.

In the sequel we will illustrate the iterative refinement of a concern model via numerical simulation, numerical solution of optimal control problems by direct optimization methods and modeling. For the first model we use, see Lesourne and Leban (1978), Feichtinger and Hartl (1986) and Will (1992):

$$\dot{X} = (1-\tau)\,P - D\,, \tag{1}$$
$$\dot{Y} = I - \delta\,(X+Y) - (1-\tau)\,P + D\,, \tag{2}$$
$$\dot{J} = e^{-rt}\,D \tag{3}$$

with $P = p\,F - \omega\,L - \rho_k\,Y - \delta\,(X+Y)$ and $F = \alpha\,(X+Y)^{\alpha_k}\,L^{\alpha_l}$. The dot denotes the derivative w. r. t. the independent variable $t \in [t_0, t_f]$. Initial time t_0 and terminal time t_f are fixed. The state variables X, Y and J denote equity capital and loan capital of the concern and the accumulation of the discounted dividends, respectively. With the output F, the profit P is the difference between sales proceeds $p\,F$ and labour costs $\omega\,L$, loan capital costs $\rho_k\,Y$, and depreciation $\delta\,(X+Y)$. τ denotes the tax rate and r denotes the notational interest on equity capital. The performance index $J(t_f)$ is to be maximized with the control functions D, I, and L, which denote dividend, investment, and number of employees. The state constraints $0 \le Y$ and $Y \le \kappa\,X$ (borrowing limit) and the control constraints $0 \le D$, $0 \le I \le I_{\max}$, and $0 \le L$ have to be fulfilled for all $t \in [t_0, t_f]$. For analytical and numerical calculations with this first concern model, see LESOURNE and LEBAN (1978), FEICHTINGER and HARTL (1986), WILL (1992) and KOSLIK et al. (1993).

Refinement of the first concern model

New direct optimization methods, e.g., the direct collocation method DIRCOL, see VON STRYK (1993), VON STRYK and BULIRSCH (1992), enable an comfortable, fast and reliable numerical solution of optimal control problems. In detail, major advantages of the direct collocation method are:

- Even non differentiable model functions, e.g., i and ρ_k in the new model, can be handled, since no explicit numerical integration of the differential equations is done, see Fig. 1.
- The large domain of convergence enables the computation of the optimal solution even with a poor initial guess.
- Although the calculation of the adjoint differential equations is not necessary, the direct collocation method DIRCOL yields accurate estimates for the adjoint variables. These estimates facilitate the use of an highly accurate indirect optimization method, e.g., the multiple shooting method.

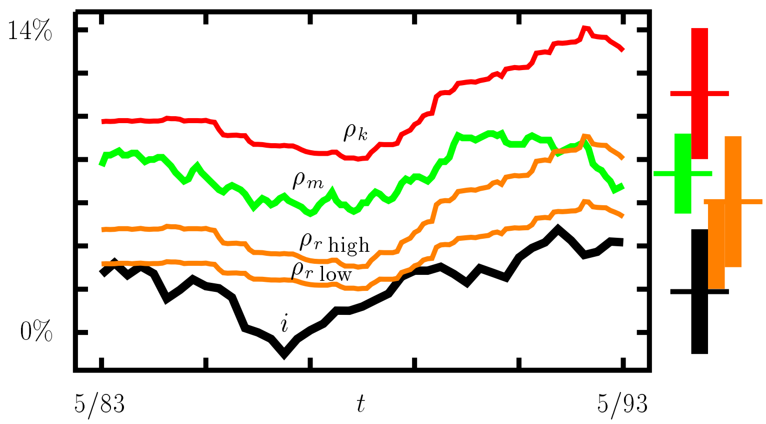

Fig. 1: Real economic functions with mean value and fluctuation for the period May 1983 to May 1993 in Germany (West): Inflation rate i (thick black curve), interest rate ρ_k for loan capital (thin dark gray curve), current yield ρ_m (thick gray curve) and two risk premium rates $\rho_{r\,\text{low}}$ and $\rho_{r\,\text{high}}$ for the equity capital in the concern (thin light gray curves).

Various refinements of the first concern model (1) – (3) lead to the new concern model:

$$\begin{aligned}
\dot{S} &= S_c\,, && (4)\\
\dot{L} &= L_c\,, && (5)\\
\dot{Y} &= Y_c\,, && (6)\\
\dot{X} &= +I + (1-\tau)\,(P - \rho_r\,X)\,, && (7)\\
\dot{X}_m &= -I + (1-\tau)\,X_m\,\rho_m\,, && (8)\\
\dot{X}_r &= (1-\tau)\,\rho_r\,X\,, && (9)\\
\dot{d} &= -d\,\ln(1+i) && (10)
\end{aligned}$$

with

$$F = \alpha\,(X+Y)^{\alpha_k}\,L^{\alpha_l}\,,$$

and

$$P = \frac{1}{d}\,(p\,(F - S_c) - \sigma\,S - \omega\,L) - \rho_k\,Y - \delta\,(X+Y)\,.$$

Equations (4) and (5) represent the level of stocks $S \in [S_{\min}, S_{\max}]$ and number of employees $L \geq 0$ (new controls $S_c \in [S_{c,\min}, S_{c,\max}]$ and $L_c \in [L_{c,\min}, L_{c,\max}]$). The loan capital $Y \in [0, \kappa\,X]$ can be controlled via the new control $Y_c \in [Y_{c,\min}, Y_{c,\max}]$ replacing the former I, see (2) and (6). The owner of the equity capital $X \geq 0$ in the concern holds a remaining part $X_m \geq 0$ of his capital in alternative capital assets, e.g., fixed-interest stocks. With the help of the investment $I \in [I_{\min}, I_{\max}]$, the capital flow between X and X_m can be directed, see (7) and (8). The demand for a risk premium by the owner of the equity capital $X \geq 0$ in the concern is modeled in Eq. (9). For numerical calculations with real economic data or realistically modeled economic cycles, see BREITNER et al. (1993) and KOSLIK et al. (1993), it is necessary, to calculate an **exact** discounting function $d(t)$ for a **variable** inflation $i(t)$. The derivation of the Eq. (10) and the initial condition $d(t_0) = 1$ for $d(t)$ can be found in KOSLIK et al. (1993). For the investigation of realistically modeled economic cycles it is comfortable, to add the equation

$$\dot{k}_p = \frac{2\pi}{k_l(t)} \tag{11}$$

for the position k_p in an economic cycle with the initial condition $k_p(t_0) = k_{p,0}$. The duration k_l of the economic cycle can be chosen even discontinuous. With the help of k_p, the economic functions can be modeled as trigonometric functions, e.g., $i(t) := i_m + i_v \sin k_p(t)$. All the economic parameters and function, e.g., τ and δ have been determined carefully. The owner of the capital $(X + X_m)$ and the management of the concern try to maximize the total profit,

$$Z = X\,(t_f) + X_m\,(t_f) + (1-\tau)\,p\,\frac{S(t_f)}{d(t_f)} \quad \longrightarrow \quad \max\,!$$

with respect to the control functions S_c, L_c, Y_c and I.

The derivation of the Eqs. (4) – (11), the related initial conditions and the state and control constraints for the design of a realistic concern model requires some iterations of simulation, optimization and modeling. Optimal solutions for various economic settings have to be calculated numerically for all preliminary models. These solutions must be compared to well known real phenomenona in micro- and macroeconomy, see, e.g., HEINEN (1985). A further refinement of the model – equations, boundary conditions, constraints or model functions – is required, as long as unrealistic solutions are obtained. The final, complex and very realistic concern model can be found in KOSLIK et al. (1993).

Numerical results

Various numerical calculations with the realistic concern model have shown its validity. The optimal solutions gain insight into the optimal management of a concern on the one hand and can help to understand macroeconomic phenomenona on the other hand. The numerical results include optimal solutions and optimal limit cycles

- for the real data of the inflation i, the interest rate for loan capital ρ_k and the current yield ρ_m, see Fig. 1 and Fig. 2;
- for realistic economic cycles including non-constant cycle duration $k_l \in [3\,\text{years}, 9\,\text{years}]$;
- different planing horizons t_f (1 year, 3, 5 and 10 years);
- for the best possible duration ${k_l}^*(t)$ of the economic cycle related to the initial conditions (best case analysis), see Fig. 3;
- for the worst possible duration $k_{l*}(t)$ of the economic cycle related to the initial conditions (worst case analysis).

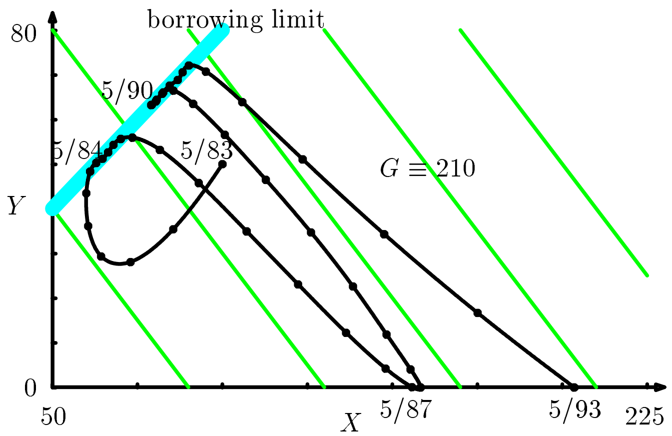

Fig. 2: History of loan capital Y versus equity capital X (black curve) for the moderate risk premium $\rho_{r\,\text{low}}$. The dots on the curve mark the collocation nodes of the direct collocation method DIRCOL. The borrowing limit (thick gray line) and lines of constant joint capital $G := X + Y$ (thin gray lines) are also depicted.

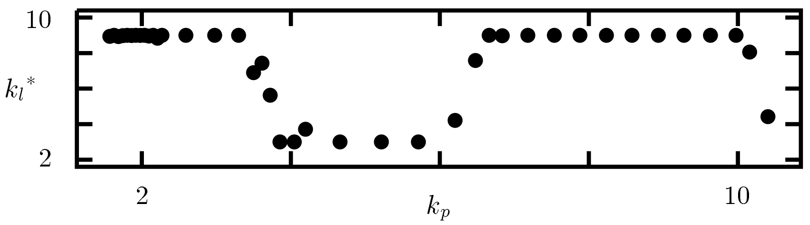

Fig. 3: Best case $k_l^* \in$ [3 years, 9 years] versus k_p (dots): Good estimate with the help of the direct collocation method DIRCOL.

Many of the calculated optimal solutions and optimal limit cycles can be found in KOSLIK et al. (1993). Our future research in this area is devoted to the numerical calculation of optimal solutions with indirect optimization methods, e.g., the multiple shooting method, too. Furthermore, the application of differential game theory, see, e.g., BREITNER et al. (1993), is planed for the handling of unknown, future economic data.

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